On a theorem of Kulikov and Dieudonné.

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§ 1. Introduction.

KULIKOV was the first to give a necessary and sufficient condition for an abelian p-group of arbitrary power to be a direct sum of cyclic groups [5].[1]) As a generalization of this criterion, DIEUDONNÉ obtained recently the following result [1]: Let G be an abelian p-group and B a subgroup of G such that G/B is a direct sum of cyclic groups. In order that G be a direct sum of cyclic groups, *it is necessary and sufficient that B be the union of* a countable ascending sequence of subgroups with elements of bounded height (relative to G). The special case where $B=G$ is the result mentioned of KULIKOV.

In a previous paper [2] I have given a criterion of different kind for the decomposibility of an abelian p-group into the direct sum of cyclic groups, from which KULIKOV's criterion can easily be deduced. In a more recent paper I generalized this result, obtaining thus a necessary and sufficient condition for an abelian p-group of arbitrary power to be fully decomposable [3]. A group is said to be fully decomposable if it can be represented as a direct sum of directly indecomposable groups. It is known that among the abelian torsion groups only the groups $C(p^m)$ $(0 \leqq m \leqq \infty)$ — i. e. the cyclic and quasicyclic p-groups — are directly indecomposable [4], [7].[2])

In what follows I shall prove a theorem (see Theorem 1 in § 2) which is a common generalization of DIEUDONNÉ's theorem and of my results mentioned above. As special cases of this theorem, several other theorems of the theory of groups can be obtained (see § 3). In § 4 we also prove that this theorem cannot be sharpened.

In his paper DIEUDONNÉ adds several interesting remarks to his theorem. E. g. he mentions that if B and G/B are direct sum of cyclic groups, the same does not hold in general for G. Moreover, by an ingenious counter-example he also shows that it is not sufficient to require the former con-

[1]) The numbers in brackets refer to the Bibliography given at the end of the paper

[2]) For notation and terminology see the end of this §.

ditions, even if G contains no elements of infinite height. Our theorem answers also to the question of what additional requirement must be fulfilled in order that G be a direct sum of cyclic groups. This requirement is the existence of a *principal system* of elements in B (relative to G).

Since an abelian torsion group is the direct sum of uniquely determined p-groups, our result can be extended in an obvious way to torsion groups of arbitrary power.

The notations and the terminology used are the following. By capital letters we denote groups or some systems of group elements, by the letters $x, a, b, \ldots, g$ group elements, while the other small Latin letters are reserved for rational integers, in particular p for a prime number. The Greek letter ν may range over an arbitrary not necessarily ordered set of indices. In what follows, by a group we shall mean always an additively written abelian group with more than one element. A subgroup generated by the elements $a, b, \ldots$ of a group is denoted by $\{a, b, \ldots\}$. A group every element of which is of finite order, is called a *torsion group.* It is well known that a torsion group may be represented as the direct sum of its uniquely determined *primary components,* each of which is a p-group, i. e. a group containing only elements of p-power order. We denote the order of a group element a by $O(a)$. The *height* in G of an element a of the p-group G is defined as follows. An element $a \neq 0$ of the p-group G is said to have the height $h = H(a)$ if the equation $p^n x = a$ is solvable in G for $n \leqq h$, but not for $n > h$. We define $H(a) = \infty$ if $p^n x = a$ has a solution $x \in G$ for each natural number n. We emphasize that $H(a)$ is defined only for $a \neq 0$.[3]) An important role is played in our investigations by the concepts of elements of inner resp. outer infinite height. We say that an element a with $H(a) = \infty$ of the p-group G is an *element of inner infinite height* if $p^t x = a$ has a solution $x \in G$ of infinite height for each natural number t. In the contrary case, when there exists a t for which $p^t x = a$ admits only solutions $x \in G$ of finite height, we call a an *element of outer infinite height.* We remark that if G is the direct sum of its subgroups $B_1, B_2, \ldots$ and $g = b_1 + b_2 \cdots$ $(b_\nu \in B_\nu)$, then evidently $H(g) \leqq H(b_\nu)$ $(\nu = 1, 2, \ldots)$ holds.

The elements $a_1, \ldots, a_n$ of the group G are called *independent* if $r_1 a_1 + \cdots + r_n a_n = 0$ implies $r_1 a_1 = \cdots = r_n a_n = 0$. An arbitrary set S of elements of G is independent, if every finite subset of S is independent. The independence so defined is therefore a property of finite character and consequently, by virtue of ZORN's lemma (or the equivalent lemma of TUKEY), every subset R of G contains a *maximal independent system* S. If $R = G$, we call S a maximal independent system of G. We denote by $C(p^m)$ in the

3) By the *height* of an element g of G we always mean the number $H(g)$ defined above, i. e. the height refers always to the whole group G, even when an element is, for the moment, considered as an element of some subgroup of G.

case $m < \infty$ the cyclic group of order p^m, and in the case $m = \infty$ the group of type (p^∞) or *quasicyclic group,* i. e. the group $\{a_1, a_2, \ldots\}$ defined by the relations

(1) $$a_1 \neq 0,\ pa_1 = 0,\ pa_2 = a_1, \ldots .$$

§ 2. Fully decomposable abelian p-groups.

In formulating the Theorem 1 below, we shall make use of the following

Definition. A maximal independent system P of a subgroup B of the abelian p-group G is called a *principal system* of B (relative to G), if no element of P can be exchanged for an element of a greater height of B without violating the independence of the system[3])[4]). In particular, in the case $B = G$ the system P is called a principal system of G.

Remarks. One can see that a principal system P (relative to G) of the subgroup B is subject to a condition considerably stronger than the one determining a principal system of B (no respect being paid to the imbedding of B into the group G). By a principal system we therefore shall always mean one relative to G, even if this is not explicitely stated.

Each element of finite height of a principal system P is of order p, for an element $a \in P$ of order p^k $(k > 1)$ would be exchangeable for the element $p^{k-1}a$ of greater height. On the other hand, an element of infinite height can obviously be exchanged for an element of order p. Consequently, in what follows we may always assume that a *principal system contains only elements of order p.*

Now we state the following

Theorem 1. *An abelian p-group G is fully decomposable — i. e. it decomposes into the direct sum of cyclic and quasicyclic groups — if and only if*

1) *G contains no elements of outer infinite height,*

2) *there exists a subgroup B of G having a principal system (relative to G), and*

3) *G/B is a direct sum of cyclic groups.*

Proof. The necessity of conditions 1), 2) and 3) can easily be verified. In fact, let us suppose that G is the direct sum of cyclic and quasicyclic groups C_ν, and let be $B = G$. Then 1) and 3) obviously hold.

In order to prove the validity of 2), we choose an element a_ν of order p from each direct summand C_ν of $B = G$. We show that the set of all a_ν's is a principal system P of G. Indeed, P is a maximal independent system

[4]) Of course, elements of infinite height are considered to be of greater height than any element of finite height.

in G; moreover, for an arbitrary element $b \in G$ of order p^k a relation of the form

$$p^{k-1} b = r_1 a_1 + \cdots + r_n a_n$$

holds (with a suitable notation for the elements of P), showing that $a_1, \ldots, a_n$ are the only elements of P one of which can be replaced by b without violating the independence of the system P. Since $H(b) \leqq H(p^{k-1} b) \leqq H(a_i)$ $(i = 1, \ldots, n)$, no element of P can be exchanged for an element of a greater height of G.

In order to prove the sufficiency of conditions 1)—3) let us consider an arbitrary abelian p-group G without elements of outer infinite height. Moreover, let B be a subgroup of G containing a principal system $P = (a_\nu)$ (relative to G) and suppose that $G/B = \bar{G}$ is a direct sum of cyclic groups:

$$\bar{G} = \sum_{k=1}^{\infty} \bar{A}_k$$

the $\bar{A}_k$'s being direct sums of groups $C(p^k)$ (the case $\bar{A}_k = 0$ not excluded). Let us denote with D_k the subgroup of G, the elements of which are exactly those belonging to the cosets relative to B of the group $\bar{A}_1 + \cdots + \bar{A}_k$. By D_0 we mean $D_0 = B$. We state that *for the height*[3]) *of an arbitrary element g_k not contained in B of the group D_k an estimate $H(g_k) \leqq k-1$ holds for each* $k > 0$. For if $g_k \in D_k$, $g_k \notin B$, then $\bar{g}_k \neq 0$ where by $\bar{g}_k$ we denote the coset of G (relative to B) containing the element g_k. Obviously the equation $p^n \bar{x} = \bar{g}_k$ can be solvable in $\bar{G} = G/B$ only with $n \leqq k-1$, and consequently the same holds for the equation $p^n x = g_k$ in G.

Now we extend the given principal system $P = P_0$ of B to a maximal independent system P^* of G, as follows. We adjoin a maximal independent system of elements $\in D_1$ of order p (and of height 0) such that the resulting system P_1 shall be a maximal independent system of D_1. In an analogous way we extend P_1 to a maximal independent system P_2 of D_2 by adjoining a maximal system of elements $\in D_2$ of order p and of height 1 ("1-layer"), then a maximal system of elements $\in D_2$ of order p and of height 0 ("0-layer"). We proceed likewise in constructing successively the systems $P_3, P_4, \ldots$, each P_n being a maximal independent system of D_n. Clearly *the union P^* of all systems $P = P_0, P_1, P_2, \ldots$ is a maximal independent system of G, the elements of wich are all of order p.*

Now we are going to show that the group G is fully decomposable. If a_ν is an element of infinite height of P^*, then from 1) we infer the existence of an infinite system of elements $a_\nu^{(1)}, a_\nu^{(2)}, \ldots$ in G, such that

$$p a_\nu^{(1)} = a_\nu, \; p a_\nu^{(2)} = a_\nu^{(1)}, \ldots$$

hold. Then (see (1))

$$\{a_\nu^{(1)}, a_\nu^{(2)}, \ldots\} = C_\nu = C_\nu(p^{m_\nu}) \qquad (m_\nu = \infty)$$

is a quasicyclic subgroup of G containing a_ν. On the other hand, if for an element $a_\nu \in P^*$

$$H(a_\nu) = h_\nu < \infty$$

holds, then we determine an element $c_\nu \in G$ such that

(2) $$p^{h_\nu} c_\nu = a_\nu.$$

In this case let

$$\{c_\nu\} = C_\nu = C_\nu(p^{m_\nu}) \qquad (m_\nu = h_\nu + 1).$$

Now we state that G is the direct sum of the groups C_ν. As a matter of fact, the independence of the system P^* guarantees that the subgroup K of G generated by the C_ν's is the *direct sum* of these groups:

$$K = \sum C_\nu.$$

Therefore only the fact $K = G$ remains to be proved.

Before proving this, we make the following remark: *A relation*

(3) $$pg = d_1 + \cdots + d_m \qquad (g \in G, d_i \in C_i)$$

implies $d_i = pd_i'$ $(d_i' \in C_i, i = 1, \ldots, m)$. For let us suppose the contrary. Then there exists a relation

(4) $$pg = r_1 c_1 + \cdots + r_k c_k \qquad (g \in G)$$

for which $p \nmid r_j$ $(j = 1, \ldots, k)$. In fact, each d_i in the right member of (3) which can be expressed in the form $d_i = pd_i'$ $(d_i' \in C_i)$ — and for a quasicyclic C_i this is always the case — disappears by introducing $g' = g - d_i'$. Now, among the elements $a_1, \ldots, a_k$ corresponding to the elements $c_1, \ldots, c_k$ in (4) let a_1 be one of maximal height;

$$h_1 = H(a_1) \geqq H(a_j) \qquad (j = 1, \ldots, k).$$

Thus, by virtue of (2), (4) implies

$$a' = p^{h_1+1} g = r_1 a_1 + \cdots.$$

This means that $H(a') > H(a_1)$ which contradicts the following statement: *if $a_1, \ldots, a_j$ are arbitrary element of P^* and*

(5) $$a' = s_1 a_1 + \cdots + s_j a_j$$

then

(6) $$H(a') \leqq H(a_i) \qquad (i = 1, \ldots, j).$$

This can be verified as follows. We may assume that the elements $a_1, \ldots, a_j$ are so ordered in (5) that

(7) $$a_i \in D_m \text{ implies } a_{i-t} \in D_m \qquad (t = 1, \ldots, i-1).$$

If (6) is not true, then let i be the maximal index among $1, \ldots, j$ such that

(8) $$H(a') > H(a_i).$$

Hence

(9) $$H(a_{i+u}) \geqq H(a') > H(a_i) \qquad (u = 1, \ldots, j-i).$$

Consequently, by (5) and (7),

$$g' = a' - (s_{i+1} a_{i+1} + \cdots + s_j a_j) = s_1 a_1 + \cdots + s_i a_i \in D_m$$

(contained in D_m together with a_i) with $H(g') > H(a_i)$ (see (8), (9)) g' may replace the element $a_i \in D_m$ without violating the independence of P_m, which contradicts the construction of the system P_m (more exactly the maximality of the "$H(g')$-layer" of P_m) in the case $m > 0$, resp. the definition of $P = P_0$ in the case $m = 0$.[5]) This completes the proof of our previous remark.

Now we can easily prove that $K = G$. Suppose that K is a proper subgroup of G. Then there exists an element $g \in G$ such that

$$g \notin K, \quad pg \in K \tag{10}$$

i. e. a relation (3) holds. Thus $d_i = pd_i'$ ($d_i' \in C_i$, $i = 1, \ldots, m$) and so we have

$$p(g - d_1' - \cdots - d_m') = 0.$$

Therefore

$$g' = g - d_1' - \cdots - d_m' \tag{11}$$

— as an element $\neq 0$, see (10) — is an element of order p, and consequently, by the maximal property of the system P^*, g' is a linear combination of some a_ν's, i. e. $g' \in K$. Hence (11) implies $g \in K$ which contradicts (10), completing the proof of Theorem 1.

§ 3. Applications.

From Theorem 1 easily follow several well-known theorems. These we obtain by specialization in two directions: putting $B = G$, and, on the other hand, requiring that G shall not contain elements of infinite height.

In case $B = G$ one obtains from Theorem 1 the following result of mine [3]:

Theorem 2. *An abelian p-group G is fully decomposable if and only if G contains no element of outer infinite height, and there exists a principal system of elements in G.*

Supposing that G contains no elements of infinite height, one obtains from Theorem 2 a former result of mine, stating when an abelian p-group is the direct sum of cyclic groups [2]:

Theorem 3. *An abelian p-group G containing no element of infinite height is a direct sum of cyclic groups if and only if there exists a principal system of elements in G.*

For corollaries of Theorems 2 and 3 see [2] and [3].

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[5]) We should like to call attention to the fact that in the case $m = 0$ the element g' lies, by (7), in $D_0 = B$.

In the special case if B contains no element of infinite height, from Theorem 1 we obtain

Theorem 4. *An abelian p-group G is a direct sum of cyclic groups if and only if there exists in G a subgroup B without elements of infinite height*[7]) *containing a principal system (relative to G), and G/B is a direct sum of cyclic groups.*

Remark. Dieudonné gives an ingenious example of an abelian p-group without elements of infinite height, which has a subgroup B such that both B and G/B are direct sums of cyclic groups without G itself being a direct sum of cyclic groups. Theorem 4 states the additional requirement which must be fulfilled in order that no such situation shall occur.

From Theorem 4 there follows Dieudonné's theorem:

Theorem 5. *Let G be an abelian p-group and B a subgroup of G such that G/B is a direct sum of cyclic groups. Then G is a direct sum of cyclic groups if and only if B is the union of a countable ascending sequence*

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$$

of subgroups such that each A_n contains an element of maximal height $h_n < \infty$.[8])

The necessity of the conditions in Theorem 5 is evident. In order to prove their sufficiency, we remark that by Theorem 4 it suffices to construct a principal system P of elements in B (relative to G) as follows. Choose a maximal independent system of elements $\in A_1$ of height h_1, adjoin a maximal system of elements $\in A_1$ of height $h_1 - 1$, and so on, so that the resulting system P_1 shall be a maximal independent system of A_1.[9]) In an analogous way we extend P_1 to a maximal independent system P_2 of A_2. We proceed likewise in constructing successively the systems $P_3, P_4, \ldots$ each P_n being a maximal independent system of A_n. No we assert that the union P of all systems $P_1, P_2, \ldots$ is a principal system of B (relative to G). Clearly, *P is a maximal independent system of B, and the elements of P are all of order p.* Hence for an arbitrary element g of order p^k of G a representation

$$a' = p^{k-1}g = s_1 a_1 + \cdots + s_j a_j \tag{12}$$

holds, $a_1, \ldots, a_j$ being suitable elements of P. This equation shows that $a_1, \ldots, a_j$ are the only elements of P one of which may be replaced by g without violating the independence of the system P. On the other hand, we have seen in § 2 that equation (12) implies the validity of (6), and, a fortiori, the validity of

$$H(g) \leqq H(a_i) \qquad (i = 1, \ldots, j).$$

This shows that P is a principal system of B (relative to G) completing so the proof of Theorem 5.

The special case of Theorem 5 where $B=G$ is a theorem of KULIKOV:

Theorem 6. *An abelian p-group G is the direct sum of cyclic groups if and only if G is the union of a countable ascending chain*

$$A_1 \subseteq A_2 \subseteq \cdots \subset A_n \subseteq \cdots$$

of its subgroups such that each A_n contains an element of maximal height $h_n < \infty$.[3])

§ 4. Concluding remarks.

Finally we show that our Theorem 1 is not capable of further sharpening.

First we remark that condition 1) in Theorem 1 cannot be omitted. In order to show this consider the following group $G=\{a_0, a_1, a_2, \ldots\}$ (Prüfer [6]) defined by the relations

$$a_0 \neq 0,\ pa_0 = 0,\ a_0 = pa_1 = p^2a_2 = \cdots = p^n a_n = \cdots.$$

This group is not fully decomposable since it contains an element of outer infinite height, viz. a_0. Conditions 2) and 3) of Theorem 1, however, hold for this group G, for it is not hard to see that the system

$$a_0,\ a_1 - pa_2,\ pa_2 - p^2a_3,\ \ldots,\ p^{n-1}a_n - p^n a_{n+1},\ \ldots$$

is a principal system of $B=G$.

Next we show that condition 2) of Theorem 1 cannot be omitted. Consider the complete direct sum of the cyclic groups $\{b_n\} = C(p^n)$ $(n=1,2,3,\ldots)$, i. e. the set of all "vectors"

$$c = \langle m_1 b_1, m_2 b_2, \ldots, m_n b_n, \ldots \rangle$$

which are added component-wise. The elements of finite order of this group form an abelian p-group G wich contains no element of infinite height. Recently T. SZELE has shown in a very simple way that this group G cannot be decomposed into a direct sum of cyclic groups [8]. Therefore G is not fully decomposable, although it fulfils condition 3) of Theorem 1 with $B=G$.

Finally we show that condition 3) of Theorem 1 cannot be omitted. Moreover we are going to prove that condition 3) cannot be replaced by the weaker requirement: G/B is a direct sum of cyclic and quasicyclic groups. For there exists an abelian p-group G without elements of infinite height containing a subgroup B with a principal system (relative to G) such that G/B is fully decomposable, although G itself is not fully decomposable. Such a group is e. g. the group G mentioned in the preceding paragraph. Since this group is not fully decomposable and it contains no element of infinite height, we have only to show that G possesses a subgroup B such that G/B is fully decomposable and there exists in B a principal system (relative to G). This can easily be proved with the aid of on important theorem of KULIKOV, which asserts that an arbitrary abelian p-group contains a subgroup B such that B is a direct sum of cyclic groups, G/B is a direct sum of quasicyclic groups and the height of each element of B relative to B is identical with that relative to G [5]. Now it is not hard to see that this last property of the subgroup B implies that it contains a principal system relative to G. For let B be the direct sum of the cyclic groups $\{c_\nu\}$ where c_ν is an element of order p^{k_ν}. We show that the set P of all elements $a_\nu = p^{k_\nu - 1} c_\nu$ is a principal system of B (relative to G). Clearly P is a maximal independent system in B. Moreover for an arbitrary element $g \in B$ of order p^k a relation

$$p^{k-1} g = r_1 a_1 + \cdots + r_n a_n$$

holds (with a suitable notation for the elements of P), showing that $a_1, \ldots, a_n$ are the only elements of P one of which can be replaced by g without violating the independence of P. Since

$$H(g) \leqq H(p^{k-1} g) \leqq H(a_i) \qquad (i = 1, \ldots, n),$$

where $H(x)$ denotes the common height of an element $x \in B$ relative to B and relative to G, no element of P can be exchanged for an element of greater height of B. Therefore P is a principal system of B, as stated above.

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